Quantum tunneling, dynamical symmetry, and quantum revival

Ju-Yong Shin and Hai-Woong Lee
Department of Physics, Korea Advanced Institute of Science and Technology, Taejon, 305-701, Korea
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Quantum tunneling dynamics of a periodically driven symmetric double-well system is studied within the framework of the Floquet theory. The role that a dynamical symmetry of the system plays in connection with tunneling is analyzed. The analysis shows that, in order to fully describe tunneling exhibited by a system with a dynamical symmetry, one should look at wave functions at times corresponding to integral multiples of the period of the driving force plus one-half period. The analysis shows also that tunneling can be understood to occur as a result of the quantum revival, a phenomenon well known in quantum optics.

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I. INTRODUCTION

Recent investigations [1–5] of quantum tunneling in a double-well system in the presence of a periodic driving force have led to a deeper understanding of the quantum dynamics of a periodically driven system. The approach based on the Floquet theory [6–8] in particular has proved most useful, as it provides a convenient framework in which to discuss tunneling. For example, coherent tunneling between regular islands recently observed [1] can be understood in terms of Floquet states localized on the islands [2]. Within the Floquet formalism, the tunneling rate is determined simply by quasienergy splittings of the Floquet states involved. Thus, it can readily be understood that application of the driving force can lead to an enhancement [1] or reduction [3] of the tunneling rate depending on the amplitude and frequency of the force.

From a fundamental viewpoint, tunneling can be viewed as a quantum effect related to symmetries of the system. The first such realization led to an identification of the process known as dynamical tunneling [4]. The occurrence of quantum barrier tunneling can also be predicted solely from an argument based on a dynamical symmetry of the system being considered [5].

In this paper we discuss quantum tunneling that occurs in a symmetric double-well system (e.g., a particle in a symmetric double square-well potential or a Duffing double-well oscillator) driven by a periodic force using the Floquet approach. We wish in particular to address the question concerning the relation between quantum barrier tunneling and dynamical symmetry, which was briefly considered in the past [5] but has not been fully exploited.

Two main findings of the present investigation are the following:

(1) For a system with a dynamical symmetry, Floquet states possess either odd or even dynamical parity, and consequently a state corresponding to spatial reflection of the initial state is reached at certain odd-half-integer multiples of the period of the driving force. Thus, quantum tunneling in a system with a dynamical symmetry cannot be fully described by looking only at the wave functions at times of integral multiples of the driving period. A complete tunneling is achieved rather at times corresponding to integral multiples of the period plus one-half period.

(2) As long as quasienergies form a discrete rational set, even if the initial state consists of a large number of Floquet states, a state corresponding to spatial reflection of the initial state is reached at a time (which turns out to be an odd-half-integer multiple of the driving period) when the oscillating Floquet states (almost) come back in phase with each other. Thus, tunneling in a system with a dynamical symmetry is not restricted to initial states forming a superposition of two eigenstates of opposite parity; it can also occur when initial states consist of a larger number of Floquet states. At a time twice the tunneling time, the oscillating Floquet states again come back in phase with each other, this time to reproduce the initial state. This phenomenon of tunneling and recurrence is analogous to the quantum revival [9] observed in optical systems.

II. THEORY

Let us consider a particle of mass m in a symmetric double-well potential driven by a periodic force of period T. The Hamiltonian

$$H(q,p,t)=\frac{p^2}{2m}+V(q)+qF(t) \tag{1}$$

is of course periodic in time,

$$H(q,p,t+T)=H(q,p,t), \tag{2}$$

since $F(t+T)=F(t)$. In addition we assume that the force satisfies the relation $F(t+T/2)=-F(t)$, which holds, for example, if the force varies sinusoidally in time. This relation together with the assumed symmetry of the potential $[V(-q)=V(q)]$ yield

$$H(-q,p,t+T/2)=H(q,p,t), \tag{3}$$

i.e., the Hamiltonian for our system possesses a dynamical symmetry. An immediate consequence of Eq. (3) is that each Floquet state $\chi_n(q,t)$ is either odd or even with respect to the transformation, $t\rightarrow t+T/2$ and $q\rightarrow -q$; i.e.,

$$\chi_n(q,t+T/2)=\pm\chi_n(-q,t), \tag{4}$$

where the plus (minus) sign is to be applied to a state of even (odd) dynamical parity.

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According to the Floquet theory, if the initial state $\psi(q,t=0)$ is expanded in terms of Floquet states $\chi_n(q)\equiv\chi_n(q,0)=\chi_n(q,NT)(N=1,2,3,\ldots)$ as

$$\psi(q,t=0)=\sum_n c_n\chi_n(q), \tag{5}$$

then the state $\psi(q,t)$ at a later time $t=NT$ is given by [2,8]

$$\psi(q,NT)=\sum_n c_n\chi_n(q)e^{-iN\epsilon_n T/\hbar}. \tag{6}$$

Making use of Eq. (4) we can also write $\psi(q,t)$ at $t=(N-1/2)T$ as

$$\psi(q,(N-\tfrac{1}{2})T)=\sum_n (\pm c_n)\chi_n(-q)e^{-i(N-1/2)\epsilon_n T/\hbar}. \tag{7}$$

Equation (7) constitutes the basic relation for our discussion of tunneling. In particular, we note that, if we find an integer N_0 such that

$$(N_0-\tfrac{1}{2})(\epsilon_n-\epsilon_1)T/\hbar=2\pi m_n \tag{8}$$

(m_n is an arbitrary integer) for all ϵ_n's corresponding to the Floquet states χ_n having the same dynamical parity as χ_1, and

$$(N_0-\tfrac{1}{2})(\epsilon_n-\epsilon_1)T/\hbar=2\pi(m_n+\tfrac{1}{2}) \tag{9}$$

(m_n is an arbitrary integer) for all ϵ_n's corresponding to the Floquet states χ_n having the opposite dynamical parity to χ_1, we have from Eq. (7)

$$\psi(q,(N_0-\tfrac{1}{2})T)=\pm e^{-i(N_0-1/2)\epsilon_1 T/\hbar}\sum_n c_n\chi_n(-q), \tag{10}$$

where the plus (minus) sign should be chosen if the state χ_1 has an even (odd) dynamical parity. In order to discuss tunneling, we assume that the initial state given by Eq. (5) consists of a wave packet localized in one of the two wells of the potential. Equation (10) then indicates that the state at $t=(N_0-\frac{1}{2})T$ is a wave packet localized in the other well, because $|\psi(q,(N_0-\frac{1}{2})T)|^2=|\psi(-q,0)|^2$. Since Eq. (10) holds exactly as long as N_0 satisfies Eqs. (8) and (9), a possibility of a complete tunneling is suggested by Eq. (10).

To be specific, we consider a simple case when the initial wave packet localized in one well consists of two Floquet states, i.e., when

$$\psi(q,0)=c_{n_1}\chi_{n_1}(q)+c_{n_2}\chi_{n_2}(q). \tag{11}$$

The state at $t=(N_0-1/2)T$ is then given, according to Eq. (10), by

$$\psi(q,(N_0-\tfrac{1}{2})T)=\pm e^{-i(N_0-1/2)\epsilon_{n_1}T/\hbar}[c_{n_1}\chi_{n_1}(-q)+c_{n_2}\chi_{n_2}(-q)], \tag{12}$$

where N_0 can be chosen to be the smallest integer that satisfies

$$(N_0-\tfrac{1}{2})(\epsilon_{n_2}-\epsilon_{n_1})T/\hbar=2\pi m \tag{13}$$

(m is an arbitrary integer) if the two states χ_{n_1} and χ_{n_2} have the same dynamical parity, and

$$(N_0-\tfrac{1}{2})(\epsilon_{n_2}-\epsilon_{n_1})T/\hbar=2\pi(m+\tfrac{1}{2}) \tag{14}$$

(m is an arbitrary integer) if χ_{n_1} and χ_{n_2} have opposite dynamical parities. The tunneling time is obviously given by $\tau=(N_0-1/2)T\simeq 2\pi\hbar/|\epsilon_{n_2}-\epsilon_{n_1}|$.

It should be noted that earlier analyses of tunneling involving two Floquet states are based largely upon their symmetric and antisymmetric combinations [2,3]. One usually considers a situation in which the initial wave packet is given, for example, by a symmetric combination

$$\psi(q,0)=\frac{1}{\sqrt{2}}[\chi_{n_1}(q)+\chi_{n_2}(q)] \tag{15}$$

and looks for the time t at which the wave function is represented by an antisymmetric combination. In fact Eq. (6) indicates that, at $t=N_1T$, where N_1 is given by

$$N_1(\epsilon_{n_2}-\epsilon_{n_1})T/\hbar=2\pi(m+\tfrac{1}{2}) \tag{16}$$

(m is an arbitrary integer), the wave function is indeed given by an antisymmetric combination

$$\psi(q,N_1T)=\pm e^{-N_1\epsilon_{n_1}T/\hbar}\frac{1}{\sqrt{2}}[\chi_{n_1}(q)-\chi_{n_2}(q)]. \tag{17}$$

If the Floquet states $\chi_{n_1}(q)$ and $\chi_{n_2}(q)$ are given, respectively, by symmetric and antisymmetric functions of q peaked in both wells, then Eq. (15) represents a wave packet localized in the left well while Eq. (17) represents the same wave packet localized in the right well. We emphasize that, in order for $\psi(q,N_1T)$ of Eq. (17) to represent a wave packet tunneled through a barrier, the two Floquet states involved are required to be symmetric and antisymmetric functions of q, each possessing peaks in both the left and right wells. Tunneling based on Eq. (12) on the other hand is more general in that Eq. (12) is a direct consequence of the symmetry of the system. All that is required is that the initial state given by Eq. (11) be a wave packet localized in one well. There is no requirement on the characteristics of each individual Floquet state.

III. EXAMPLE

As an illustration we consider a particle in a double square-well potential of width $2a$ with a central barrier of width $2b$ and height V_0, i.e.,

$$V(q)=\begin{cases}\infty, & |q|>a\\ 0, & b<|q|\leqslant a\\ V_0, & |q|\leqslant b.\end{cases} \tag{18}$$

The particle is driven by a sinusoidal force of period T and amplitude F_0, so that the Hamiltonian for the system is

$$H=\frac{p^2}{2m}+V(q)+qF_0\cos\left(\frac{2\pi t}{T}\right). \tag{19}$$

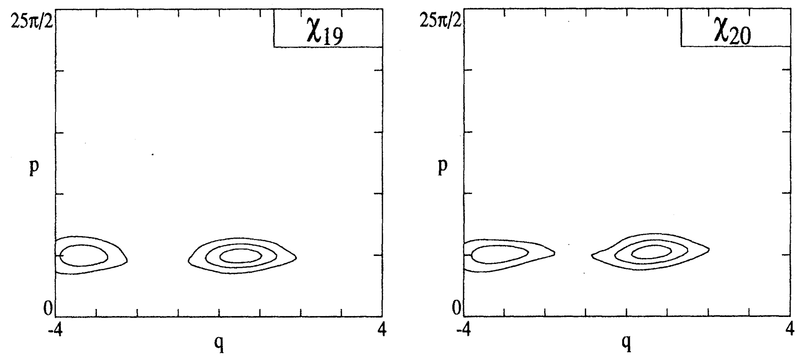

FIG. 1. Husimi phase-space plots of two Floquet states χ_{19} and χ_{20} for the driven double-well system with $a=4, b=0.01, V_0=500, F_0=2.5$, and $T=1$. Contours are drawn in steps of 0.01.

This Hamiltonian not only is periodic but also possesses a dynamical symmetry. Our computation has been performed with parameters chosen as $a=4, b=0.01, V_0=500, F_0=2.5$, and $T=1$. With these parameters, the number of eigenstates with energy less than $V_0=500$ is ~ 80.

In Fig. 1 we show Husimi plots [10] of two Floquet states χ_{19} and χ_{20}. These two states correspond, respectively, to the 18th and 19th excited states in the limit $F_0 \rightarrow 0$, whose energy eigenvalues lie well below V_0. They look almost identical in the phase-space plots with peaks in the left well and near the center, but χ_{19} is a symmetric function of q while χ_{20} is an antisymmetric function. They also possess opposite dynamical parities. As an initial state we choose a symmetric combination of these two Floquet states

$$\psi(q,0)=\frac{1}{\sqrt{2}}[\chi_{19}(q)+\chi_{20}(q)], \tag{20}$$

and compute the time development. Husimi plots of the wave function $\psi(q,t)$ at some selected times are shown in Fig. 2. The initial wave function given by Eq. (20) is seen to be a wave packet localized in the left well. It can also be seen that $\psi(q,T)$ is not much different from $\psi(q,0)$. In fact, for the system being considered, we have in general $|\psi(q,t+T)| \approx |\psi(q,t)|$, because, according to our numerical computation, the quasienergies satisfy

$$\frac{\pi}{(\epsilon_{19}-\epsilon_{20})T/\hbar} \simeq 98.84, \tag{21}$$

i.e., $(\epsilon_{19}-\epsilon_{20})T/\hbar \simeq \pi/98.84 \ll 1$. Application of this inequality to Eq. (6) immediately yields $|\psi(q,T)| \approx |\psi(q,0)|$.

It is clear from Fig. 2 that the initial wave packet has tunneled through the central barrier at $t=98.5T$. Substitution of Eq. (21) into Eq. (14) yields indeed that Eq. (14) is satisfied approximately with $m=0$ and $N_0=99$. The tunneling time is thus given by $(N_0-1/2)T=98.5T$, in agreement with the observation based on Fig. 2. It is also interesting to note that Eq. (16) is satisfied approximately with $m=0$ and $N_1=99$. Thus, the wave function at $t=99T$ is given approximately by an antisymmetric combination of χ_{19} and χ_{20}. In this case the antisymmetric combination does not yield a wave packet localized deep on the right side of the right well

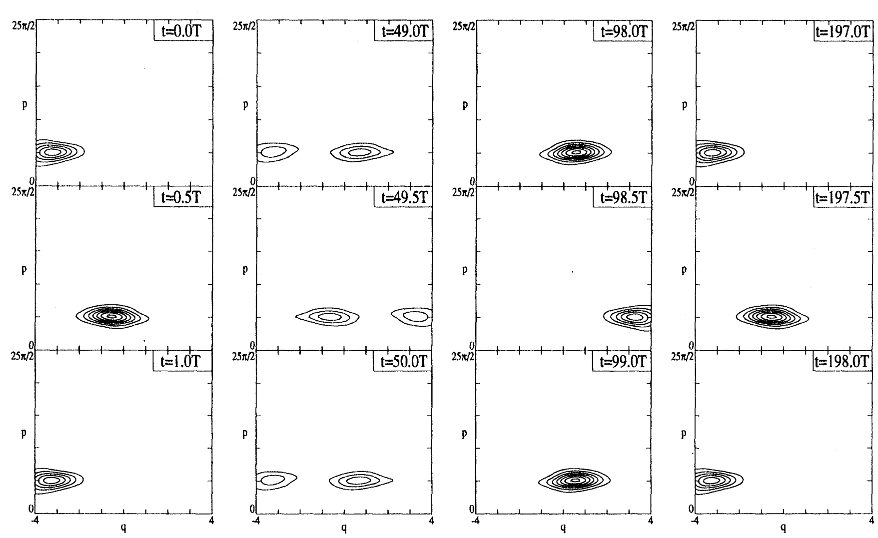

FIG. 2. Husimi phase-space plots of the wave function $\psi(q,t)$ at some selected times. The initial state is chosen to be a symmetric combination of χ_{19} and χ_{20}. Parameters are the same as in Fig. 1.

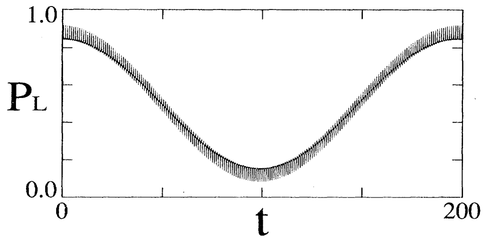

FIG. 3. The probability $P_L(t)$ to find the particle in the left well at time t vs time t. Parameters are the same as in Fig. 1.

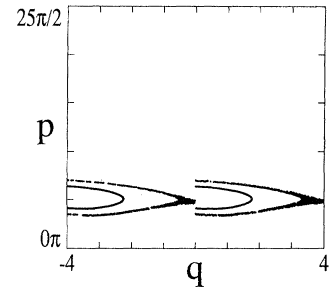

FIG. 4. Classical Poincaré map of the driven double-well system. Parameters are the same as in Fig. 1.

(see Fig. 2 at $t=99T$), because neither the Floquet state χ_{19} nor χ_{20} has a peak centered in that neighborhood.

As further evidence of tunneling we plot in Fig. 3 the probability $P_L(t)$ of finding the wave packet in the left well at time t, defined as

$$P_L(t)=\int_{-a}^{0}|\psi(q,t)|^2dq. \tag{22}$$

The probability $P_L(t)$ is seen to decrease to its minimum value $t\simeq 98.5T$ and increase back to its intial value at $t\simeq 197T$. This oscillatory tunneling with a period $197T$ continues in the absence of dissipation. The fast oscillations of period T seen on a small scale can be understood as arising from the classical oscillation of the wave packet in the absence of the central barrier. Since the barrier is very thin ($b=0.01$), there is a finite probability for the wave packet to tunnel through the barrier and oscillate between the two wells at $q=\pm a=\pm 4$ with period $T=1$.

IV. DISCUSSION

A word about our choice of an initial wave packet is in order. In general, a wave packet localized near an elliptic fixed point of resonance consists of a small number of localized Floquet states, whereas that localized near a separatrix consists of many delocalized Floquet states [11–13]. Shown in Fig. 4 is the classical Poincaré map of our double-well system with the same parameters as before. Clearly, the Floquet states χ_{19} and χ_{20} shown in Fig. 1 reveal the structure of quantum nonlinear resonance [11,14] similar to the classical structure. It can easily be expected that their linear combination should produce a wave packet localized in one of the elliptic fixed points of Fig. 4. On the other hand, if a wave packet localized in the region away from the elliptic fixed points of the resonance is to be constructed, many Floquet states need to be superposed. In this case many quasienergies are involved and it gets more difficult to find an integer N_0 and a set of integers m_n that satisfy Eqs. (8) and (9). Nevertheless, as long as the initial wave packet consists of a finite number of Floquet states having discrete values of quasienergy, an integer N_0 that at least approximately satisfies Eqs. (8) and (9) should exist, and thus tunneling should eventually occur. In this respect quantum tunneling is analogous to the phenomenon of quantum revival observed in optical systems [9].

Finally, we wish to comment briefly on the question of what happens to tunneling if the dynamical symmetry of the system is broken slightly, e.g., if the width or depth of the two potential wells is not exactly the same. We have shown earlier that a driven particle in a double square well behaves more regularly when the potential is symmetric than when it is not [15]. It should be emphasized that what we describe in the present work is tunneling of a coherent nature, which occurs as a direct consequence of symmetry. The phenomenon of tunneling is thus more clearly defined if the system behaves more regularly, i.e., if the system symmetry is less severely broken.

In conclusion we have shown that quantum tunneling exhibited by a periodically driven system in a symmetric potential can be viewed as a necessary consequence of a dynamical symmetry of the system. The degree of tunneling depends generally on quantitative relations between quasienergy splittings of Floquet states that constitute the initial wave packet. Tunneling can be observed at times corresponding to certain odd-half-integer multiples of the period of the driving force and can be understood to occur as a result of the quantum revival mechanism well known in quantum optics.

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